Mathematical Model for Simulated Projectile Hit Detection

1 Definitions and Coordinate System

We work in an inertial reference frame with the gun's pivot point as the origin at the instant of trigger pull t_0. All vector quantities are in $\mathbb{R}^3$.

Known Constants

- $f \in \mathbb{R}^+$: camera focal length (pixels)
- $S \in \mathbb{R}^+$: known physical diameter of the target
- (u_0, v_0): camera principal point (image center, pixels)
- $v_p \in \mathbb{R}^+$: projectile muzzle speed (scalar)
- $R : \mathbb{R}^+ \to \mathbb{R}^+$: projectile spread radius as a function of distance traveled, monotonically increasing up to a maximum range
- $\boldsymbol{g} \in \mathbb{R}^3$: gravitational acceleration

Barrel State at t_0 (from IMU)

- $\hat{\boldsymbol{b}} \in S^2$: unit vector along the barrel axis
- $\boldsymbol{\omega} \in \mathbb{R}^3$: angular velocity of the barrel
- $\boldsymbol{r}_m \in \mathbb{R}^3$: position vector from pivot to muzzle tip
- $M(t) \in \mathrm{SO}(3)$: rotation matrix from barrel frame to inertial frame at time t, determined by integrating $\boldsymbol{\omega}$

Target State at t_0 (estimated)

- $\boldsymbol{r}_0 \in \mathbb{R}^3$: position in the inertial frame
- $\boldsymbol{v} \in \mathbb{R}^3$: velocity in the inertial frame
- $\boldsymbol{a} \in \mathbb{R}^3$: acceleration in the inertial frame (assumed constant)

2 Target Observation Model

At each camera frame with timestamp t_i, the target is detected with pixel centroid (u_i, v_i) and apparent diameter s_i (pixels).

Distance from Apparent Size

$$d_i = \frac{S\,f}{s_i} \tag{1}$$

Ray Direction in Barrel Frame

The pixel offset from the image center defines a ray direction in the barrel (camera) frame. Let $\Delta u_i = u_i - u_0$ and $\Delta v_i = v_i - v_0$. The unit ray direction is:

$$\hat{\boldsymbol{e}}_i = \frac{1}{\sqrt{\Delta u_i^2 + \Delta v_i^2 + f^2}} \begin{pmatrix} \Delta u_i \\ \Delta v_i \\ f \end{pmatrix} \tag{2}$$

Position in Barrel Frame

$$\boldsymbol{r}_i^b = d_i \, \hat{\boldsymbol{e}}_i \tag{3}$$

Transformation to Inertial Frame

$$\boldsymbol{r}_i = M(t_i) \, \boldsymbol{r}_i^b \tag{4}$$

where $M(t_i)$ is obtained from the IMU-measured angular velocity history.

3 State Estimation

A Kalman filter $\mathcal{K}$ ingests the sequence of inertial-frame observations and produces the maximum-likelihood target state estimate at trigger time t_0:

$$\mathcal{K} : \{\boldsymbol{r}_i\}_{i=1}^{N} \longmapsto \left(\hat{\boldsymbol{r}}_0, \ \hat{\boldsymbol{v}}, \ \hat{\boldsymbol{a}}\right) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \tag{5}$$

The process model internal to $\mathcal{K}$ assumes constant acceleration, i.e., the state transition for interval Δt is

$$\begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{v} \\ \boldsymbol{a} \end{pmatrix}_{t+\Delta t} = \begin{pmatrix} I & \Delta t \, I & \frac{1}{2}\Delta t^2 \, I \\ 0 & I & \Delta t \, I \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{v} \\ \boldsymbol{a} \end{pmatrix}_t \tag{6}$$

where each block entry is 3×3 and I is the identity.

4 Projectile Initial Conditions

The projectile departs from the muzzle at t_0 with initial conditions:

$$\boldsymbol{p}_0 = \boldsymbol{r}_m \tag{7}$$

$$\boldsymbol{V} = v_p \, \hat{\boldsymbol{b}} + \boldsymbol{\omega} \times \boldsymbol{r}_m \tag{8}$$

The second term accounts for the transverse velocity imparted to the projectile by the barrel's swing.

5 Trajectories After Trigger Pull

For elapsed time $\tau > 0$ after the trigger pull:

Target Position

$$\boldsymbol{r}(\tau) = \hat{\boldsymbol{r}}_0 + \hat{\boldsymbol{v}}\,\tau + \tfrac{1}{2}\,\hat{\boldsymbol{a}}\,\tau^2 \tag{9}$$

Projectile Center

$$\boldsymbol{p}(\tau) = \boldsymbol{p}_0 + \boldsymbol{V}\tau + \tfrac{1}{2}\,\boldsymbol{g}\,\tau^2 \tag{10}$$

Separation Vector

Define the relative state quantities:

$$\boldsymbol{\delta}_0 = \hat{\boldsymbol{r}}_0 - \boldsymbol{p}_0 \qquad \text{(initial separation)} \tag{11}$$

$$\boldsymbol{\delta}_v = \hat{\boldsymbol{v}} - \boldsymbol{V} \qquad \text{(relative velocity)} \tag{12}$$

$$\boldsymbol{\delta}_a = \hat{\boldsymbol{a}} - \boldsymbol{g} \qquad \text{(relative acceleration)} \tag{13}$$

The separation between target and projectile center at time τ is then:

$$\boldsymbol{\Delta}(\tau) = \boldsymbol{r}(\tau) - \boldsymbol{p}(\tau) = \boldsymbol{\delta}_0 + \boldsymbol{\delta}_v\,\tau + \tfrac{1}{2}\,\boldsymbol{\delta}_a\,\tau^2 \tag{14}$$

6 Projectile Travel Time

We define the intercept time τ^* as the moment when the projectile's expanding sphere passes through the target's along-track position. Specifically, τ^* is when the separation vector has no component along the projectile's velocity—i.e., the target lies in the plane normal to $\boldsymbol{V}$ passing through the projectile center.

Remark. This is an intercept condition, not a closest-approach condition. The closest approach of target and projectile center would instead require $\boldsymbol{\Delta}(\tau)\cdot\dot{\boldsymbol{\Delta}}(\tau) = 0$, which yields a different (cubic) equation. The intercept formulation is the physically appropriate one: we ask whether the target is within the projectile's cross-sectional disc at the moment the projectile passes by.

The condition is:

$$\boldsymbol{\Delta}(\tau^*)\cdot\boldsymbol{V} = 0 \tag{15}$$

Expanding:

$$\big(\boldsymbol{\delta}_0\cdot\boldsymbol{V}\big) + \big(\boldsymbol{\delta}_v\cdot\boldsymbol{V}\big)\,\tau^* + \tfrac{1}{2}\big(\boldsymbol{\delta}_a\cdot\boldsymbol{V}\big)\,\tau^{*2} = 0 \tag{16}$$

Define the scalar coefficients:

$$\alpha = \boldsymbol{\delta}_a\cdot\boldsymbol{V}, \qquad \beta = \boldsymbol{\delta}_v\cdot\boldsymbol{V}, \qquad \gamma = \boldsymbol{\delta}_0\cdot\boldsymbol{V} \tag{17}$$

Then τ^* satisfies the quadratic $\tfrac{1}{2}\alpha\,\tau^{*2} + \beta\,\tau^* + \gamma = 0$, yielding:

$$\boxed{\tau^* = \frac{-\beta \pm \sqrt{\beta^2 - 2\alpha\gamma}}{\alpha}} \tag{18}$$

We take the smallest positive root. In the special case $\alpha = 0$ (no relative acceleration along $\boldsymbol{V}$):

$$\tau^* = -\frac{\gamma}{\beta} = \frac{\boldsymbol{\delta}_0\cdot\boldsymbol{V}}{\big(\boldsymbol{V} - \hat{\boldsymbol{v}}\big)\cdot\boldsymbol{V}} \tag{19}$$

which reduces to range divided by closing speed along the line of fire.

7 Hit Condition

At time τ^*, compute the miss distance. Since $\boldsymbol{\Delta}(\tau^*) \cdot \boldsymbol{V} = 0$ by construction, $\boldsymbol{\Delta}(\tau^*)$ is entirely perpendicular to the projectile's direction of travel, so the miss distance is simply its magnitude:

$$\rho = \big\|\boldsymbol{\Delta}(\tau^*)\big\| = \big\|\boldsymbol{\delta}_0 + \boldsymbol{\delta}_v \, \tau^* + \tfrac{1}{2} \, \boldsymbol{\delta}_a \, \tau^{*2}\big\| \tag{20}$$

The projectile has traveled a distance $\ell = v_p \, \tau^*$ (to first order). The hit condition is:

$$\boxed{\rho \le R(\ell)} \tag{21}$$

That is: the perpendicular miss distance is within the projectile's spread radius at the distance traveled.

If τ^* has no positive real root (discriminant $\beta^2 - 2\alpha\gamma < 0$), or if ℓ exceeds the projectile's maximum effective range, the result is a miss.

8 Summary of Algorithm

At each frame t_i prior to trigger pull:

1. Detect the target in the image, yielding (u_i, v_i, s_i).
2. Compute d_i, $\hat{\boldsymbol{e}}_i$, and $\boldsymbol{r}_i^b$ via equations (1)–(3).
3. Transform to inertial frame: $\boldsymbol{r}_i = M(t_i) \, \boldsymbol{r}_i^b$ using IMU data.
4. Update the Kalman filter with $\boldsymbol{r}_i$.

At trigger pull t_0:

1. Read the Kalman state estimate $(\hat{\boldsymbol{r}}_0, \hat{\boldsymbol{v}}, \hat{\boldsymbol{a}})$.
2. Compute the projectile initial conditions $\boldsymbol{p}_0$ and $\boldsymbol{V}$ via equations (7)–(8).
3. Compute α, β, γ and solve for τ^* via equation (16).
4. Evaluate $\rho = \|\boldsymbol{\Delta}(\tau^*)\|$ and test $\rho \le R(v_p \, \tau^*)$.